

Lecture 23

Oracle TMs and Limits of Diagonalization

Oracle Turing Machines

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Note: Nondeterministic oracle TMs are defined similarly.

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Claim: Let $EXPCOM = \{ \langle M, x, 1^n \rangle \mid M \text{ outputs } 1 \text{ on } x \text{ within } 2^n \text{ steps} \}$. Then,

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Then exponential time TM N' for L on input x :

- Simulates N on all possible u s and replace every call to oracle on $\langle M, y, 1^n \rangle$ by simulating M on y for 2^n step.

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Suppose L has a decider N that runs in at most $k2^{n^c}$ steps.

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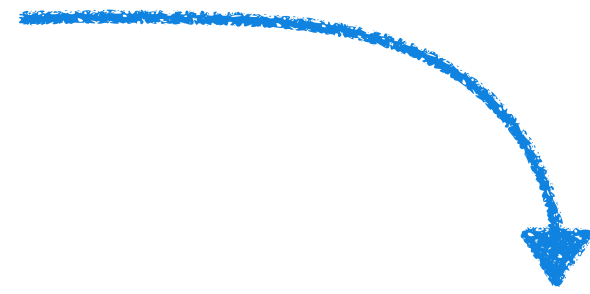
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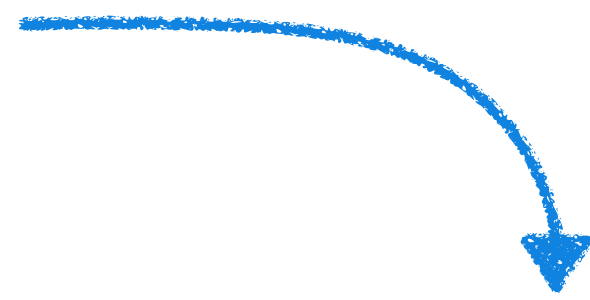
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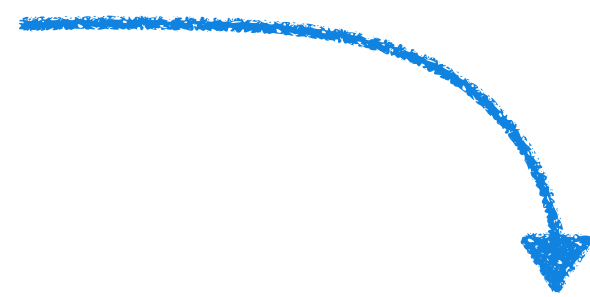


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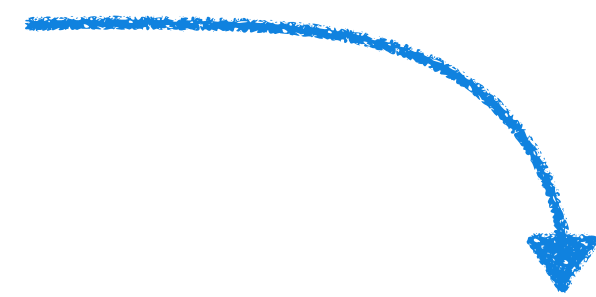


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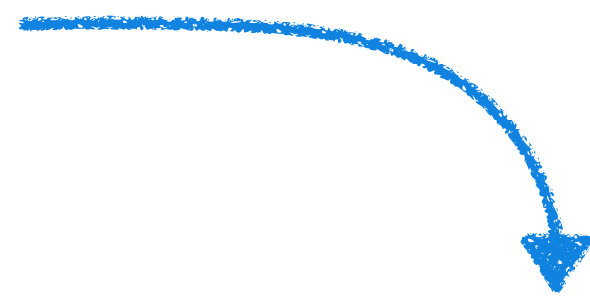


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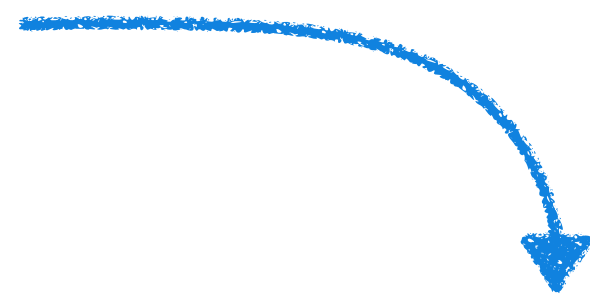


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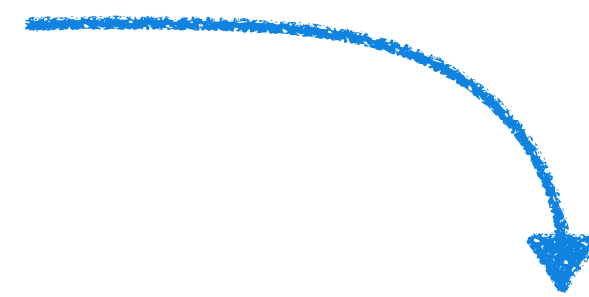


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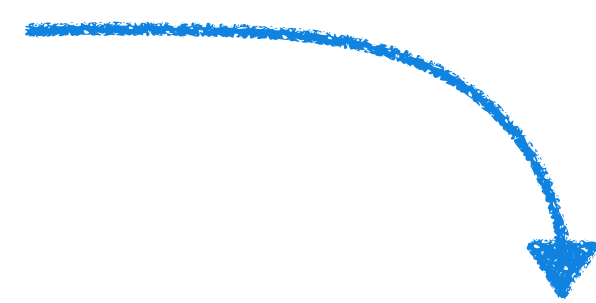
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Thus, P vs NP question cannot be settled through diagonalization "alone".

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Theorem: If $f : \mathbb{N} \rightarrow \mathbb{N}$, $g : \mathbb{N} \rightarrow \mathbb{N}$ are time-constructible functions satisfying $f(n)\log f(n) = o(g(n))$, then for any $O \subseteq \{0,1\}^*$, $\text{DTIME}(f(n))^O \subset \text{DTIME}(g(n))^O$.

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